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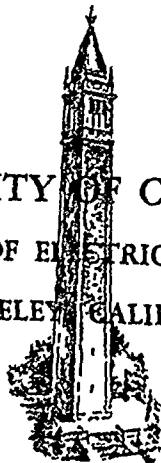ARLINGTON HALL STATION
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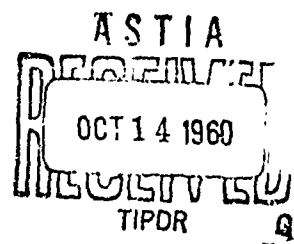## CORRELATION PROCESSES IN ANTENNA ARRAYS

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by

I. W. Linder

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CORRELATION PROCESSES IN ANTENNA ARRAYS

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I. W. Linder

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I. INTRODUCTION

The design of an antenna array whose performance can be optimized in some specific sense when the array is utilized to receive a signal buried in a general noisy medium can be accomplished through noise theory considerations and correlation techniques. In this report, the correlation coefficient of the noise voltage will be shown to relate the noise power received to the element spacing in a linear antenna array. Then, antenna arrays employing multiplicative and time averaging circuitry which implements the definition of correlation coefficient will be examined in some detail in order to determine their voltage response patterns and their resolution characteristics.

For the purpose of this analysis assume that the noise sources are spherically distributed and are statistically independent. The voltage produced on an isotropic antenna by the noise originating in ι element of solid angle dΩ can be represented by the usual Fourier series:

$$x(t) = \sum_{k=0}^{\infty} [a_k \cos(2\pi f_k t) + b_k \sin(2\pi f_k t)]$$

where

$$\lim_{T \to \infty} T \cdot E\{a_m a_n\} = 0, \qquad \text{for } m \neq n$$

$$\lim_{T \to \infty} T \cdot E\{b_m b_n\} = 0, \qquad \text{for } m \neq n$$

$$\lim_{T \to \infty} T \cdot E\{a_m b_n\} = 0$$

$$\lim_{T \to \infty} T \cdot E\{a_n^2\} = \lim_{T \to \infty} T \cdot E\{b_n^2\} = 2 W(f_n).$$

In this representation, the coefficients a_k and b_k are distributed normally with means zero; E is the expectation operator, and W(f) is the (two-sided) power spectral density induced on the antenna element by the source.

The noise voltage will be applied to the elements of a linear antenna array. Each element will be connected to an individual amplifier and phase shifter. The output of each antenna-amplifier-

phaseshifter circuit is connected to a combining network which will operate on the individual element voltages to provide an optimum output voltage from the total array. In this report, the effect of additive and multiplicative operations in the combining network will be considered.
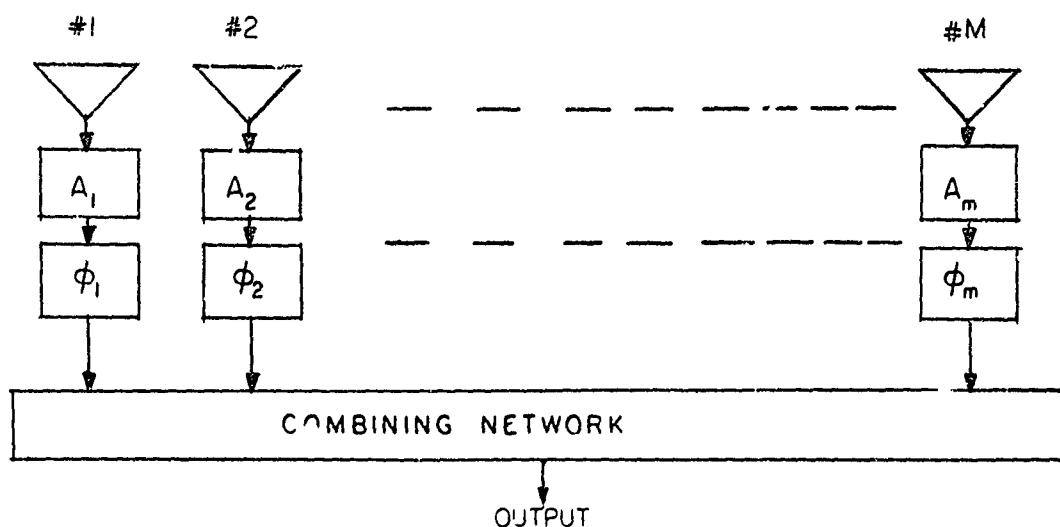
II. SIMPLE ADDING OPERATIONS IN THE COMBINING NETWORK

Consider two elements, i and j, displaced vertically in the spherical coordinate system as shown, with a distance d_{ij} between them. Then the voltage produced in one element by a point noise source located at θ_0, ϕ_0 would be displaced in time from that produced in the other element by the factor $(d_{ij}/c) \cos \theta_0$.

In general (with $n_i(t)$ representing the noise voltage induced on element i):

$$n_i(t) = \sum_{k=0}^{\infty} a_k \cos 2\pi f_k(t + \tau_i) + b_k \sin 2\pi f_k(t + \tau_i)$$

$$n_j(t) = \sum_{k=0}^{\infty} a_k \cos 2\pi f_k(t + \tau_j) + b_k \sin 2\pi f_k(t + \tau_j)$$

where τ_i and τ_j represent time displacements relative to an arbi. ry reference point.

If $n_i(t)$ is the voltage that would be produced in an isotropic element, then the voltage in an actual element would be modified by the gain in field intensity, $g(\theta, \phi)$, of the element.

$g(\theta, \phi) = \sqrt{G(\theta, \phi)}$ where $G(\theta, \phi)$ is the power gain pattern of the element.

The voltage at the output of the element-amplifier circuit could be further modified by the phase shifter. In this analysis it will be assumed that the desired signal is located at a point such that the signal voltage produced on all elements will be in phase, and the phase shifting networks can then be set on zero. Thus the voltage produced by a noise source located at θ_0, ϕ_0 will appear at the output of the antenna element-amplifier circuit as

$$N_i(t) = [\, g_i(\theta_0 \, \phi_0) \cdot A_i \cdot n_i(t) \,]$$

It is assumed that all amplifiers have the same bandwidth characteristics with center frequency f_0 and with bandwidth B. The amplitude of the gain produced by each amplifier may vary from that

of the others; the amplitude of the gain of the i-th amplifier is indicated by A_i.

By carrying out the standard analysis of the narrow band gaussian process,[1] it can be shown that the sum of M element voltages can be represented as

$$\sum_{i=1}^{M} N_i(t) = A(t) \cos \omega_0 t + B(t) \sin \omega_0 t = C(t) \cos[\omega_0 t + \phi(t)]$$

where $A(t)$ and $B(t)$ are distributed normally with

$$E\{A\} = E\{B\} = 0$$

$$E\{A(t)\,A(s)\} = E\{B(t)\,B(s)\} =$$

$$\int_{f_0 - \frac{B}{2}}^{f_0 + \frac{B}{2}} \left[2W(f) \sum_{n=1}^{M} \sum_{m=1}^{M} g_n g_m A_n A_m \cos 2\pi f(\tau_n - \tau_m) \cos 2\pi(f - f_0)(t - s) \right] df$$

$$E\{A(t)\,B(s)\} = - E\{B(t)\,A(s)\} =$$

$$\int_{f_0 - \frac{B}{2}}^{f_0 + \frac{B}{2}} \left[2W(f) \sum_{n=1}^{M} \sum_{m=1}^{M} g_n g_m A_n A_m \cos 2\pi f(\tau_n - \tau_m) \sin 2\pi(f - f_0)(t - s) \right] df$$

The amplitude of the noise voltage at the output of the system, $C(t)$, is Rayleigh distributed with parameter $\sigma^2 = E\{A^2\}$, the average noise power (unit resistance) in the output circuit.

The noise voltage which has been calculated is that contributed by a single point noise source. The total noise voltage at the output of the antenna network is then the sum of the contributions of all noise sources (this is the integral over the sphere in the case of continuously distributed noise).

The average noise power can be expressed in terms of the correlation coefficient, $R_n(\tau)$, of the noise. Assuming a continuous

distribution of noise sources, this becomes

$$N^2 = \sum_{i=1}^{M} \sum_{j=1}^{M} A_i A_j R_n(\tau_{ij})$$

where

$$R_n(\tau_{ij}) = 2 \int_{f_0 - \frac{B}{2}}^{f_0 + \frac{B}{2}} \int_{0}^{2\pi} \int_{0}^{\pi} \frac{1}{4\pi} W(f, \theta, \phi) \cos(2\pi f \tau_{ij}) \sin\theta \, d\theta \, d\phi \, df$$

The signal voltage is assumed to be produced by a signal source located on the main lobe axis and is, therefore, in phase on each of the antenna elements. So the total signal voltage in the output circuit of the antenna system is

$$S(t) = \sum_{i=1}^{M} A_i s_i(t)$$

and the signal power is

$$S^2 = \sum_{i=1}^{M} \sum_{j=1}^{M} A_i A_j R_s(0)$$

where

$$R_s(0) = \int_{f_0 - \frac{B}{2}}^{f_0 + \frac{B}{2}} 2 W_s(f) \, df$$

The average signal-to-noise power ratio in the output circuit is

$$\frac{S^2}{N^2} = \sum_{i=1}^{M} \sum_{j=1}^{M} \frac{A_i A_j R_s(0)}{A_i A_j R_n(\tau_{ij})}$$

This equation was obtained through a different, though equi-
valent, line of reasoning in a previous report.[2] Since the signal-to-
noise power ratio is a function of the correlation coefficient of the
amplified noise voltage, it can be optimized by proper choice of
spacing between the antenna elements. For a given number of elements
in the array, the signal power is fixed, and the noise power is mini-
mized by proper selection of element spacing.

III. MULTIPLICATIVE OPERATIONS IN THE COMBINING NETWORK

The combining network can be employed to multiply together
the individual element voltages. For practical effectiveness this
requires time averaging at some point in the network. This combination
of multiplication and time averaging then satisfies the definition of the
correlation coefficient:

$$R(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} n(t)\, n(t + \tau)\, dt.$$

In considering practical networks that can be used to carry out
this correlation process, two general variables arise: (1) possible
combinations of the multiplication and averaging circuits, and (2)
restriction of time averaging to a finite time interval. The first of
these will be considered for single frequency signals and for general
random signals in this section. The second will be discussed in the
next section.

The voltage response at the output of the antenna array for an
arbitrary angular location, ϕ, of a single signal source will be defined
as the directivity pattern of the array.

A. MONOCHROMATIC SIGNALS

The antenna directivity patterns which result when individual
element voltages induced by a single monochromatic source (negligible
noise) are correlated have been investigated by several researchers.[3,4]

In general, the procedure has been to recall the directivity patterns of linear additive arrays and then to demonstrate product arrays which will give equivalent directivity patterns.

The directivity patterns for linear arrays with constant element spacing, d, can be expressed by one of two series, depending on an odd or an even number of elements in the array.

1. For an array of $2n+1$ elements, the sum pattern is given by

$$P_{2n+1}(\phi) = \sum_{k=0}^{n} A_k \cos 2ku$$

where $u = (\pi d/\lambda) \sin \phi$. The array is assumed to be symmetrical with respect to its center element, A_0, $(A_i = A_{-i})$. $P_{2n+1}(\phi)$ is the amplitude gain given by the array to an input signal $E_0 \sin(\omega t + \psi)$. The center of the array is the phase reference.

2. A linear array of $2n$ elements and constant spacing, d, has a sum pattern

$$P_{2n}(\phi) = \sum_{k=1}^{n} B_k \cos(2k - 1)u$$

Again symmetry is assumed, so B_k is the common amplification of the two elements which have distance $(k - \frac{1}{2})$ d from the array center.

Berman and Clay have described an array in which pairs of signals are multiplied together and then time averaged. The output voltage is formed by multiplying together all of the resulting time averages. With this procedure, an array of $n + 1$ elements, with successive elements spaced D, 2D, 4D, $\cdots$, 2^{n-1}D from the first element, has a directivity pattern equivalent to that of a 2^n element additive array (equation (2) above) with constant element spacing 2D. In this case the product array is about one-fourth the length of the equivalent linear array. Such a product array with four elements could be represented as
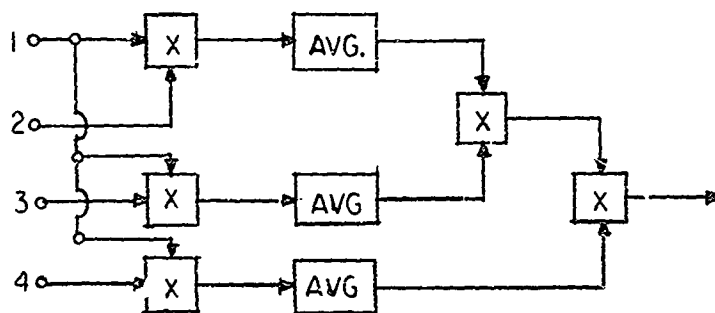
$$V_{out} \propto \prod_{i=2}^{4} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \cos \omega t \cos(\omega t - \tau_{1i}) \, dt$$

$$V_{out} \propto \prod_{i=2}^{4} \cos \omega \tau_{1i}$$

With proper spacing of the elements, this pattern is the same as that of an eight-element linear array.

A second possible product array carries out all desired multiplications before finally time averaging the product. Analysis of the directivity pattern in this case shows a smaller saving in overall length over the equivalent linear array than was the case in the first product array discussed. Again a four-element example is
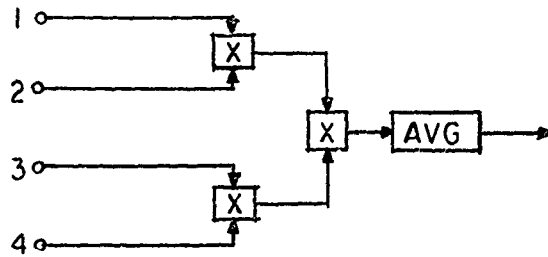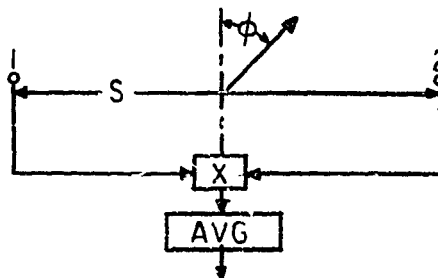
$$V_{out} \propto \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \prod_{i=1}^{4} \cos(\omega t - \tau_{1i}) \, dt.$$

With proper spacing of the elements, this pattern is the same as that of a six-element linear array.

It is also possible to start with two elements and to process the element voltages to get directional patterns which are equivalent to those of odd or even element additive arrays.

Voltage induced on element #1 $\sin(\omega t + \psi)$

Voltage induced on element #2 $\sin\left[\omega t - \left(\frac{\omega s}{c} \sin\phi\right) + \psi\right]$

Output of multiplier $Y = \sin(\omega t + \psi) \sin\left[\omega t - \left(\frac{\omega s}{c} \sin\phi\right) + \psi\right]$

After averaging $\overline{Y} = \frac{1}{2} \cos\left(\frac{\omega s}{c} \sin\phi\right)$

And, if $d = 2s$, $\overline{Y} = \frac{1}{2} \cos\left(\frac{\pi d}{\lambda} \sin\phi\right) = \frac{1}{2} \cos u$

Brown and Rowlands have shown how to use this function to synthesize a directivity pattern. Using the relation

$$T_n(\cos u) = \cos nu \qquad n = 0, 1, 2, \cdots$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind, every term in the cosine series of the desired directivity pattern can be synthesized by a suitable combination of power law devices operating on the output of the two-element correlation pair. Arithmetically, two elements can thus be made to give a directivity pattern equivalent to that of a linear additive array of any arbitrary length. In practice, noise considerations, ignored in these pattern calculations, and the presence of more than one signal source would limit the length of the equivalent linear array which could be synthesized. These points will be considered in more detail in a later section.

B. BAND-LIMITED RANDOM SIGNALS, GENERAL CASE

As with the monochromatic signals, the basic unit in the study of correlation of band-limited random signals is the two-element array. In this case the signal can be described only in statistical terms, and statistical methods can be utilized to describe the performance of the system. As before, noise will be neglected in initial considerations and will enter in a later section. Correlation in such a two-element array is directly analogous to the operation of correlation detectors, which have been described in the literature.[5] Performance of correlation detectors, however, is usually studied in terms not directly applicable to the description of the characteristics of antenna arrays.
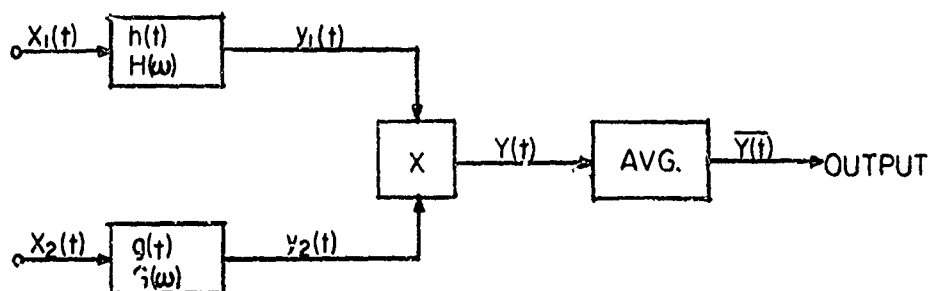
The voltages induced on the antenna elements are stationary and
ergodic random functions of time with normal distributions and are
confined to $2\pi B$, a closed interval in ω. The amplifiers are linear,
time-invariant networks with system functions $H(\omega) = |H(\omega)|\ e^{j\eta(\omega)t}$
and $G(\omega) = |G(\omega)|\ e^{j\gamma(\omega)t}$, Fourier transformable into the impulse
responses $h(t)$ and $g(t)$.

For each set of the ensemble of signal functions, the Fourier
expansion is:

$$x_1(t) = \sum_{i=1}^{N} a_i \cos(\omega_i t + \phi_i)$$

$$x_2(t) = \sum_{i=1}^{N} b_i \cos(\omega_i t + \psi_i)$$

where

     1. a_i, b_i are Rayleigh distributed with $\overline{a_i^2} = 2W_1(f_1)\,\delta f$ and $\overline{b_i^2} = 2W_2(f_i)\,\delta f$, with $W(f_i)$ as the power density over a frequency interval δf centered at f_i;

     2. ϕ_i, ψ_i have uniform distributions over $(0,\ 2\pi)$;

     3. ω_0 is the lower edge of the band of width B cps, and $\omega_i = \omega_0 + 2\pi i/T$.

Then

$$y_1(t) = \sum_{i=1}^{N} a_i\, h_i\, \cos(\omega_i t + \phi_i + \eta_i)$$

$$y_2(t) = \sum_{i=1}^{N} b_i\, g_i\, \cos(\omega_i t + \psi_i + \gamma_i)$$

After multiplication

$$Y(t) = y_1(t)\,y_2(t) = \sum_{i=1}^{N}\sum_{j=1}^{N} a_i\, b_j\, g_j\, h_i \left[\frac{1}{2}\cos(\omega_i t + \omega_j t + \phi_i + \psi_j + \eta_i + \gamma_j) + \right.$$

$$\left. \frac{1}{2}\cos(\omega_i t - \omega_j t + \phi_i - \psi_j + \eta_i - \gamma_j)\right].$$

The first cosine term describes a band of frequencies centered at twice the mid-frequency of the band-limited signal, and the second cosine term describes the dc and low frequency component of the output voltage.

     After averaging:

$$\overline{Y(t)} = \int_0^{\infty} \overline{a\,b\,g\,h}\, \cos(\phi - \psi + \eta - \gamma)\, d\omega.$$

     Now in this antenna case $x_2(t) = x_1(t)\, e^{j\omega\tau}$. So $a_i = b_i$ and $\psi_i = \phi_i + \omega_i \tau$. And if the amplifiers are assumed to have identical

phase functions and constant amplitude functions, the output is

$$\overline{Y(t)} = (G \cdot H) \int_{0}^{\infty} W(\omega) \cos \omega \tau \, d\omega = (G \cdot H) R(\tau)$$

or the correlation function. For this normally distributed signal with a rectangular frequency function (center frequency f_0)

$$\overline{Y(t)} = (G \cdot H) B \frac{\sin \pi B \tau}{\pi B \tau} \cos 2\pi f_0 \tau$$

The obvious advantage to specifying amplifiers with identical phase functions and constant amplitude functions is that this leads directly to a statement of the output of the system in terms of the correlation function of the input band-limited signal. A signal which is normally distributed over the pass band of the amplifiers can be analyzed most simply through its characteristic function. As a result of these two statements, the performance of correlation arrays with more elements can be determined in a direct manner.

Before considering more elements, however, it is useful to point out the self noise or fluctuation component of the output voltage of the multiplier in the simple two-element correlator. For a normally distributed signal source located in the principal lobe of the two-element array ($\tau = 0$), the output of the multiplier has a Chi-square distribution whose mean value is the desired array output voltage, and whose ac component is the self noise which must be minimized by time averaging. For an arbitrarily located source it was shown previously that the mean value is proportional to $R(\tau)$, the correlation coefficient. The variance is proportional to $[R(0)^2 + R(\tau)^2]$, or
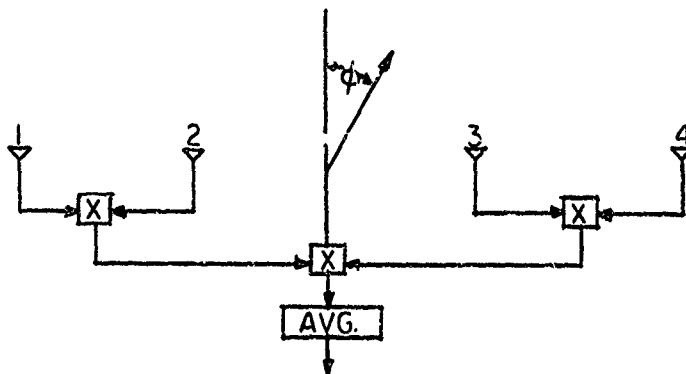
$$\sigma^2 \propto R(\tau)^2 \left[1 + \frac{R(0)^2}{R(\tau)^2} \right]$$

This variance does not go to zero. Its value decreases by $1/2$ as the signal moves from the main lobe to a position where its mean value

is zero.

As an example of the calculations of directivity patterns of correlation arrays with a greater number of elements, consider a four-element array.

The desired output voltage is simply the expected value of the product of the four element voltages.

$$\overline{Y(t)} = \overline{V_1(t) V_2(t) V_3(t) V_4(t)}$$

And, since these are normally distributed band-limited voltages, this becomes

$$Y(t) = \left[\overline{V_1(t) V_2(t)}\right]\left[\overline{V_3(t) V_4(t)}\right] + \left[\overline{V_1(t) V_3(t)}\right]\left[\overline{V_2(t) V_4(t)}\right] + \left[\overline{V_1(t) V_4(t)}\right]\left[\overline{V_2(t) V_3(t)}\right]$$

$$\propto R(\tau_{12}) R(\tau_{34}) + R(\tau_{13}) R(\tau_{24}) + R(\tau_{14}) R(\tau_{23})$$

With a rectangular frequency function, each correlation coefficient has the form

$$R(\tau_{ij}) = R(0) \frac{\sin \pi B \tau_{ij}}{\pi B \tau_{ij}} \cos 2\pi f_0 \tau_{ij}$$

And with relative element spacing chosen to make $d_{12} = D$, $d_{23} = D$, and $d_{34} = 2L$, the mean output voltage can be put in a form resembling the directivity pattern of a uniform linear array.

$$\overline{T(\tau)} \propto \left\{ \frac{1}{2} \left[\frac{\sin BX}{BX} \frac{\sin 2BX}{2BX} + \frac{\sin 2BX}{2BX} \frac{\sin 3BX}{3BX} \right] \cos 1(2f_0 X) + \right.$$

$$\frac{1}{2} \left[\frac{\sin BX}{BX} \frac{\sin 2BX}{2BX} + \frac{\sin BX}{BX} \frac{\sin 4BX}{4BX} \right] \cos 3(2f_0 X) +$$

$$\left. \frac{1}{2} \left[\frac{\sin BX}{BX} \frac{\sin 4BX}{4BX} + \frac{\sin 2BX}{2BX} \frac{\sin 3BX}{3BX} \right] \cos 5(2f_0 X) \right\}$$

where $X = (\pi D/c) \sin \phi$. For a very narrow bandwidth the coefficients of the cosine terms approach unity and the directivity pattern of this correlation array (length 4D) is the same as that of a six-element uniform linear array (length 10D). As the bandwidth increases, these coefficients influence the amplitude of the sidelobe structure of the directivity pattern. This is shown in Fig. 1 for one basic spacing, D, and for two bandwidths.

The assumption of a random signal source with a normal distribution permits a pattern analysis by the characteristic function method. The characteristic function is

$$F_x(\xi) = E\left\{ e^{j\xi_1 x_1 + j\xi_2 x_2 + \cdots + j\xi_K x_K} \right\} = \exp\left[-\frac{1}{2} \sum_{r=1}^{K} \sum_{s=1}^{K} R(r_{rs}) \xi_r \xi_s \right]$$

and the pattern will be given by the appropriate coefficient in the expansion of this function. For example, consider a simple three-element array. The directivity pattern will be given by $E\{V_1 V_2^2 V_3\}$,
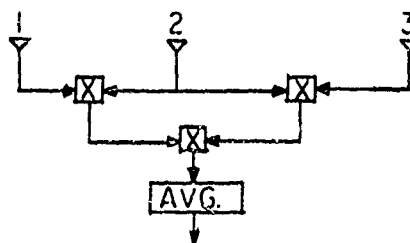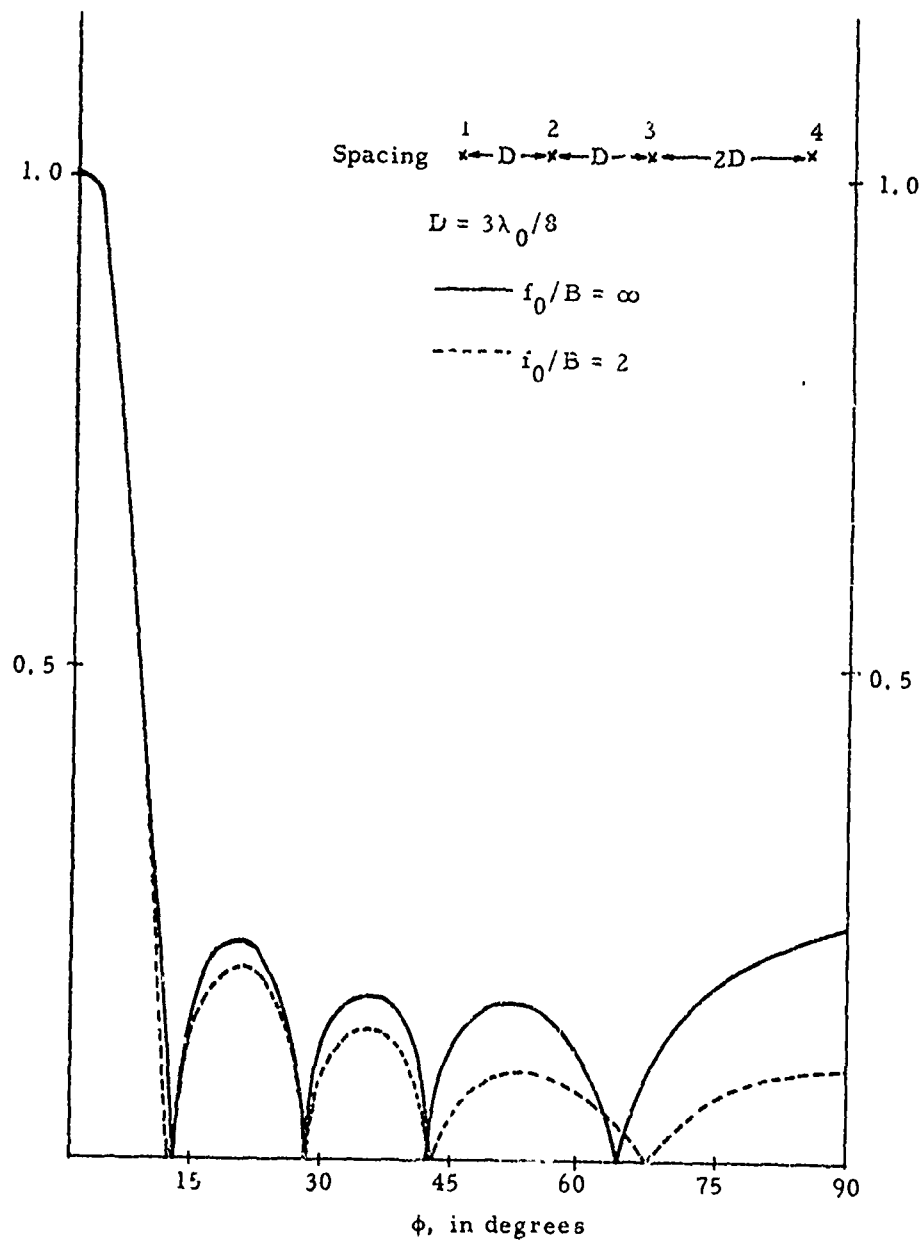
FIG. 1.—Normalized directivity pattern, four-element correlation array.

where $E\{x\}$ is the expectation operator. This will be the coefficient of $[(j\xi_1)/1 \cdot (j\xi_2)^2/2! \cdot (j\xi_3)/1]$ in the power series expansion of $\exp[-(1/2)\sum_{r=1}^{3}\sum_{s=1}^{3}R(\tau_{rs})\xi_r\xi_s]$

$$E\left\{V_1 V_2^2 V_3\right\} \propto R(0)R(\tau_{13}) + 2R(\tau_{12})R(\tau_{23})$$

If we assume a rectangular frequency function and relative element spacings $d_{12} = D$, $d_{23} = 3D$, the directivity pattern can be written

$$V_{out} \propto \left\{\left[\frac{\sin BX}{BX}\frac{\sin 3BX}{3BX}\right]\cos 2(2f_0X) + \left[\frac{\sin 4BX}{4BX} + \frac{\sin BX}{BX}\cdot\frac{\sin 3BX}{3BX}\right]\cos 4(2f_0X)\right\}$$

where $X = (\pi D/c)\sin\phi$.

The variance of the output voltage is a measure of the unwanted ac power which must be reduced by time averaging, so this quantity must be known to complete the description of a correlation array Just as in the calculation of the average output voltage, the variance can be determined by the characteristic function method, in this case by first finding $E\{(V_1 V_2^2 V_3)^2\}$. This is the coefficient of $[(j\xi_1)^2/2! \cdot (j\xi_2)^4/4! \cdot (j\xi_3)^2/2!]$ in the same power series expansion.

$$E\{(V_1 V_2^2 V_3)^2\} \propto \left[3R(0)^4 + 6R(0)^2 R(\tau_{13})^2 + 12R(0)^2 R(\tau_{12})^2 + \right.$$

$$\left. 12R(0)^2 R(\tau_{23})^2 + 24R(\tau_{12})^2 R(\tau_{23})^2 + 48R(0)R(\tau_{12})R(\tau_{23})R(\tau_{13})\right]$$

The mean and the variance of this array voltage, each normalized to unity at $(2f_0X) = 0^{\circ}$, are shown in Fig. 2.

C. BAND-LIMITED RANDOM SIGNALS, SPECIAL CASES

In the previous case no frequency restriction was imposed on the multiplication circuitry. It is apparent that as the number of antenna elements increases this operation of multiplication will impose frequency bandwidth requirements which may be difficult to meet. For each stage of multiplication added to the array circuitry, frequency requirements are doubled.
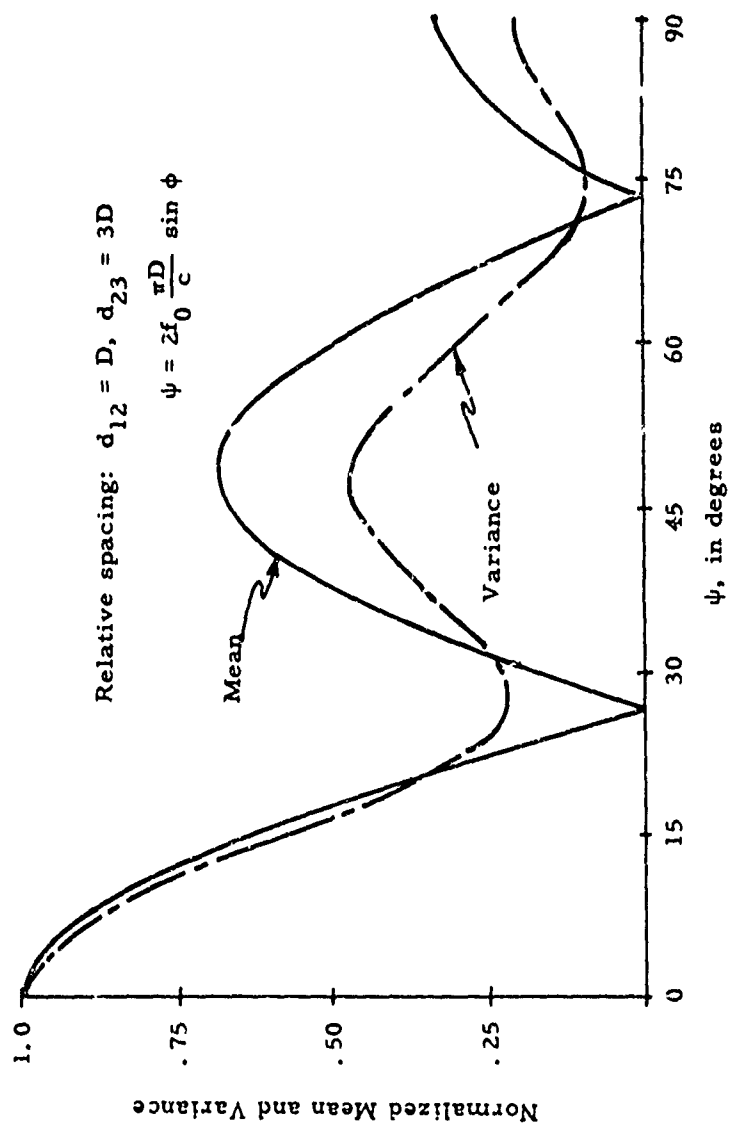
FIG. 2.—Mean and variance of output voltage, three-element correlation array.

As shown in the equations for the two element product array, the output voltage consists of a low frequency component and a component centered at $2f_0$. If this low frequency component of each product pair is selected by a low pass filter for use in subsequent stages of multiplication, the over-all frequency bandwidth requirements of the correlation circuits are kept within reasonable bounds.

In this case the directivity pattern cannot be obtained from the characteristic function but must be calculated from the low frequency term of the equation for the two-element product pair. By rejecting the double frequency component some increase in the width of the directivity pattern must be expected. It is also noted that the low frequency component is sensitive primarily to the spacing between the two elements of each pair. The spacing between different product pairs in the array influences the sidelobe level through the $(\sin nBx)/(nBx)$ coefficients. For example, consider the two-product-pair correlation array
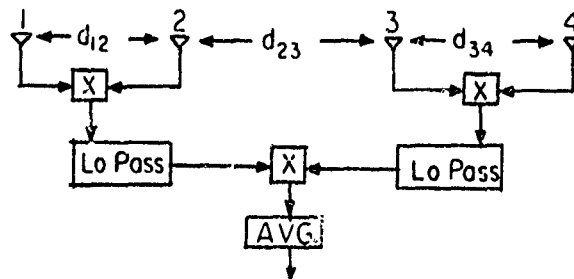
$$\overline{V_{out}} \propto \left\{ \left[\frac{\sin X_{12}B}{X_{12}B} \cdot \frac{\sin X_{34}B}{X_{34}B} + \frac{\sin X_{14}B}{X_{14}B} \cdot \frac{\sin X_{23}B}{X_{23}B} \right] \cos(X_{12} + X_{34}) 2f_0 + \right.$$

$$\left. \left[\frac{\sin X_{12}B}{X_{12}B} \cdot \frac{\sin X_{34}B}{X_{34}B} + \frac{\sin X_{13}B}{X_{14}B} \cdot \frac{\sin X_{24}B}{X_{23}B} \right] \cos(X_{12} - X_{34}) 2f_0 \right\}$$

with $X_{ij} = (\pi d_{ij}/c) \sin \phi$. Since spacing between the pairs has only a secondary effect on the directivity pattern, this can be eliminated, giving two product pairs from three elements. The directivity pattern
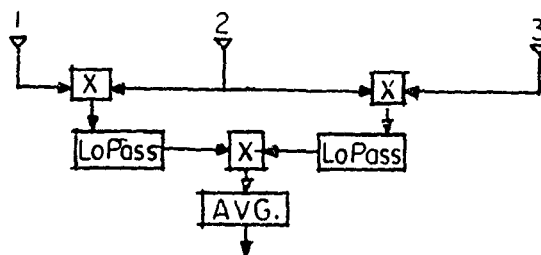
is then given by (with relative spacing D and 3D):

$$\overline{V_{out}} \propto \left\{ \left[2 \frac{\sin XB}{XB} \frac{\sin 3XB}{3XB} \right] \cos 2(2f_0 X) + \right.$$

$$\left. \left[\frac{\sin XB}{XB} \cdot \frac{\sin 3XB}{3XB} + \frac{\sin 4XB}{4XB} \right] \cos 4(2f_0 X) \right\}$$

As a comparison of the directivity patterns which result from the general correlation array utilizing all frequencies and from the low frequency correlation array, consider a four-element array under both conditions:
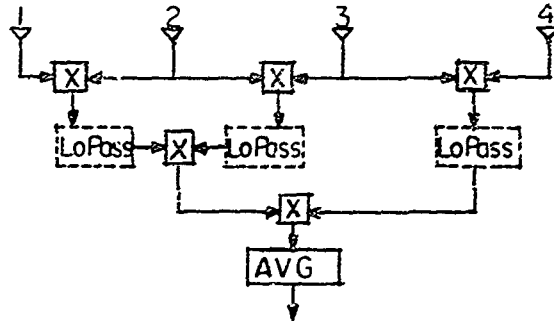
With relative spacings $d_{12} = 4D$, $d_{23} = D$, $d_{34} = 2D$, the equations
for the two cases are

1. General case

$$\overline{V_{out}} \propto [\, A \cos 9(2f_0 X) + 4B \cos 7(2f_0 X) + C \cos 5(2f_0 X) +$$

$$2D \cos 3(2f_0 X) + 2E \cos (2f_0 X)]$$

2. Low frequency case

$$\overline{V_{out}} \propto [\, F \cos 7(2f_0 X) + G \cos 5(2f_0 X) + H \cos 3(2f_0 X) +$$

$$I \cos (2f_0 X)]$$

In each case the coefficients A, B, $\cdots$ are functions of $(\sin n\,XB)/(n\,XB)$
whose values are approximately unity for narrow band situations and
become less than one only in wide-band circuits. Assuming a value
of unity for these coefficients, the directivity patterns for these two
cases are shown in Fig. 3.
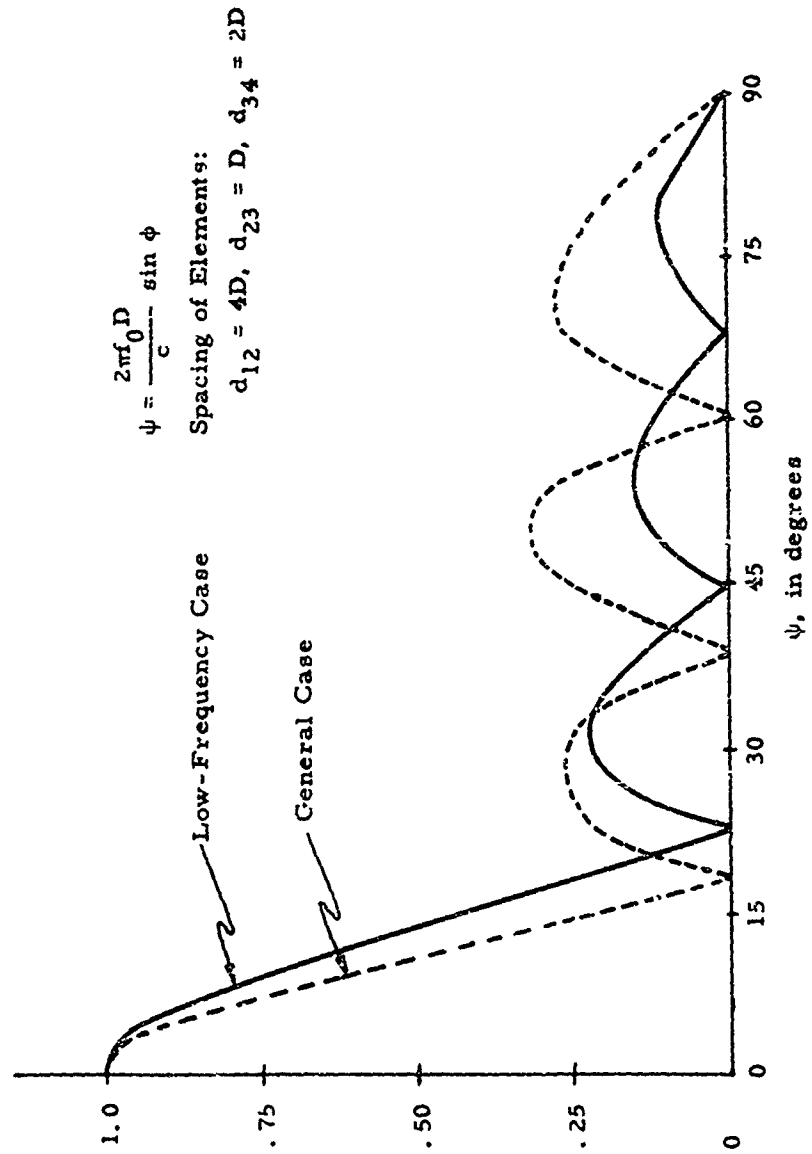
FIG. 3.—Directivity patterns, four-- ement correlation arrays.

IV. EFFECT OF FINITE AVERAGING TIME

The antenna directivity patterns in correlation arrays are exact only in the limit of infinite averaging time. In actual correlators this integration time would be a finite interval chosen as a compromise between the desired response of the system and the maximum time delay which can be tolerated. It becomes important, then, to investigate the system performance as a function of this integration time.

An expression for the effect of an averaging device can be obtained from general filter considerations. [6] If $h(t)$ is the effective weighting function of a linear measuring device, and $x(t)$ is the function to be measured, a measurement $M_x(T)$ made at time $t = T$ after $x(t)$ has been introduced at $t = 0$ can be expressed by the convolution

$$M_x(T) = \int_0^T h(u)\, x(T-u)\, du \qquad\qquad h(u) = 0,\ u < 0$$

where $(0, T)$ is the observation interval. $M_x(T)$ will vary from observation to observation, fluctuating about the expected value $\overline{M_x(T)}$ with a variance $\sigma^2 = \overline{M_x(T)^2} - [\overline{M_x(T)}]^2$.

In the general situation, $x(t)$ is at least wide sense stationary, and

$$\overline{M_x(T)} = \int_0^T h(u)\, \overline{x(T-u)}\, du$$

$$\overline{M_x(T)^2} = \int_0^T \int_0^T h(u)\, \overline{x(T-u)x(T-v)}\, h(v)\, du\, dv$$

$$= \int_0^T \int_0^T h(u)\, \psi_x(u-v)\, h(v)\, du\, dv$$

where ψ_x is the correlation coefficient of the function to be measured. For an ideal integrator $h(u) = 1/T$, $0 < u < T$. And

$$\overline{M_x(T)} = \overline{x(t)}$$

$$\overline{M_x(T)^2} = \frac{2}{T} \int_0^T \left(1 - \frac{s}{T}\right) \psi_x(s) \, ds$$

This result was obtained by Davenport[7] in an early study of correlation detectors. However, his analysis was directed to a description of signal-to-noise ratio in these detectors, while the present investigation is concerned with placing confidence intervals around the expected value of the output voltage, so the methods of utilizing the information in these equations will differ.

The information required for this statistical description of the output voltage will be the expected value and the autocorrelation coefficient of the input voltage to the time averager. The calculations will be discussed in some detail for the two-element array, and then the effect of more elements in the correlation array will be indicated.

If we consider the usual two-element array, an arbitrarily located signal source induces a band-limited random signal $V_1(t)$ on element #1 and $V_2(t) = V_1(t + \tau)$ on element #2. Then, after multiplication

$$Y(t) = V_1(t)V_2(t) = V_1(t)V_1(t+\tau)$$

$$\overline{Y(t)} = R_V(\tau)$$

$$\psi_V(s) = \overline{Y(t)Y(t+s)} = \overline{V_1(t)V_1(t+\tau)V_1(t+s)V_1(t+\tau+s)}$$

$$= R_V(\tau)^2 + R_V(s)^2 + R_V(\tau+s)R_V(\tau-s)$$

The last term of this expression for the autocorrelation coefficient of the product voltage becomes rather awkward to handle in the equations for the integrator circuit. And as more elements are added to the array, the corresponding terms in their product equations increase the complexity of the calculations. Since the interest here is in the change in integration time required as the number of elements in the correlation array increases, an approximation for the correlation coefficient will be used

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which will permit straightforward calculation of the integration characteristics.

If the signal source is located in the principal lobe ($\tau = 0$), the equations for the product voltage become

$$\overline{Y(t)} = R_V(0)$$

$$\psi_V(s) = R(0)^2 + 2R(s)^2$$

$$= R(0)^2(1 + 2\rho(s)^2)$$

where $\rho(s)$ is the normalized autocorrelation coefficient; $\rho(s) = R(s)/R(0)$. Then for an arbitrarily located source, the approximate product voltage equations are

$$\overline{Y(t)} = R_V(\tau)$$

$$\psi_V(s) \approx R_V(\tau)^2(1 + 2\rho(s)^2)$$

This approximation is good for positions near the principal lobe axis and fails for positions that make $R_V(\tau)$ quite small ($R_V(\tau) \rightarrow 0$).

For the usual rectangular band-limited voltage, $\rho(s) = (\sin \pi Bs / \pi Bs) \cos 2\pi f_0 s$, and the equations for the ideal integrator are

$$\overline{M_Y(T)} = R_V(\tau)$$

$$\overline{M_Y(T)^2} = R_V(\tau)^2 \frac{2}{T} \int_0^T \left(1 - \frac{s}{T}\right)\left(1 + 2 \frac{\sin^2 \pi Bs}{(\pi Bs)^2} \cos^2 2\pi f_0 s\right) ds$$

$$\overline{M_Y(T)^2} = R_V(\tau)^2 \frac{2}{T} \int_0^T \left(1 - \frac{s}{T}\right)\left[1 + \left(\frac{\sin \pi Bs}{\pi Bs}\right)^2 + \left(\frac{\sin \pi Bs}{\pi Bs}\right)^2 \cos 2\pi (2f_0)s\right] ds$$

The solution of this integral can be expressed in two parts, the first depending on the low frequency components of the product voltage, the second determined by the double frequency component.

$$M_Y(T)^2 = R_V(\tau)^2[1 + m_{1f}^2 + m_{2f}^2]$$

Or, in terms of the variance at the output of the integrating circuit,

$$\sigma^2 = R_V(\tau)^2 \left[m_{1f}^2 + m_{2f}^2 \right]$$

where

$$m_{1f}^2 = \left[-\frac{1-\cos 2\pi BT}{(\pi BT)^2} + \frac{2}{\pi BT} Si(2\pi BT) - \frac{1}{(\pi BT)^2} Cin(2\pi BT) \right]$$

$$\approx \frac{1}{BT} \text{ for } BT \gg 1.$$

$$m_{2f}^2 = \left[\frac{Si(2\pi BT)}{\pi BT} - \frac{Si(2k\pi BT)}{\pi BT} + \frac{(k+1)Si(2k+2)\pi BT}{2\pi BT} + \frac{(k-1)Si(2k-2)\pi BT}{2\pi BT} + \right.$$

$$\frac{\sin^2 \pi BT}{(\pi BT)^2} \cos^2 k\pi BT - \frac{Cin(2\pi BT)}{2(\pi BT)^2} - \frac{Cin(2k\pi BT)}{2(\pi BT)^2} + \frac{Cin(2k+2)\pi BT}{4(\pi BT)^2} +$$

$$\left. \frac{Cin(2k-2)\pi BT}{4(\pi BT)^2} \right]$$

where $k = (4f_0)/B$, $Si(x)$ is the sine integral, and $Cin(x) = \int_o^x (1-\cos v/v) \, dv$

The standard deviation of the voltage at the output of the inte-
gration circuit as a function of the integration time, T, is shown in
Fig. 4. In the limit as $T \to 0$, the variance is $2 R_V(\tau)^2 = 2\overline{M_V(T)}^2$, as
would be expected for the Chi-square distribution. For voltage samples
observed without integration, a 95 per cent confidence interval on the
output voltage extends from zero to $3.84 \overline{M_V(T)}$. As integration time
increases, the confidence interval becomes smaller, and observed
voltage samples would cluster more and more closely about the expected
value, $\overline{M_V(T)}$. Figure 5 shows this narrowing of the 95 per cent
confidence interval as integration time increases.

It should be noted that the double frequency component of vari-
ance decays rapidly compared to the low frequency component. A
correlator whose integration time is long compared to $1/f_0$ may still
have an output voltage with an appreciable variation because of this
low frequency variance. If the input circuits of the integrator rejected

Two-Element Array
Ideal Correlator
Single Signal Source

$\bar{Y}$: Expected Output Voltage
Band-Limited Random Signal of Bandwidth B

Double Frequency
Component

Low Frequency
Component

$2\bar{Y}^2$

$\frac{1}{2}\bar{Y}^2$

Variance, σ^2

BT
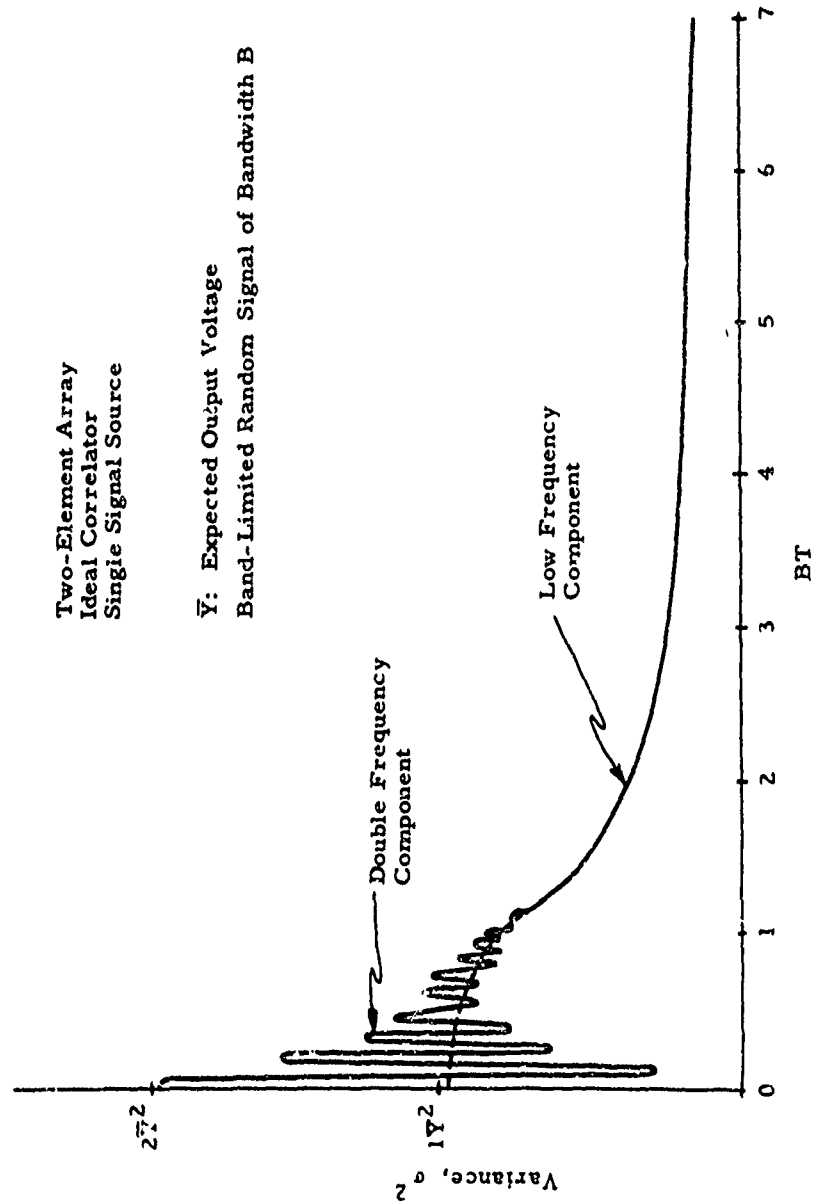
FIG. 4.—Variance of output voltage as a function of integration time, T.

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Two-Element Array
Single Signal Source

$\bar{Y}$: Expected Output Voltage

Bandlimited Random Signal
Bandwidth, B
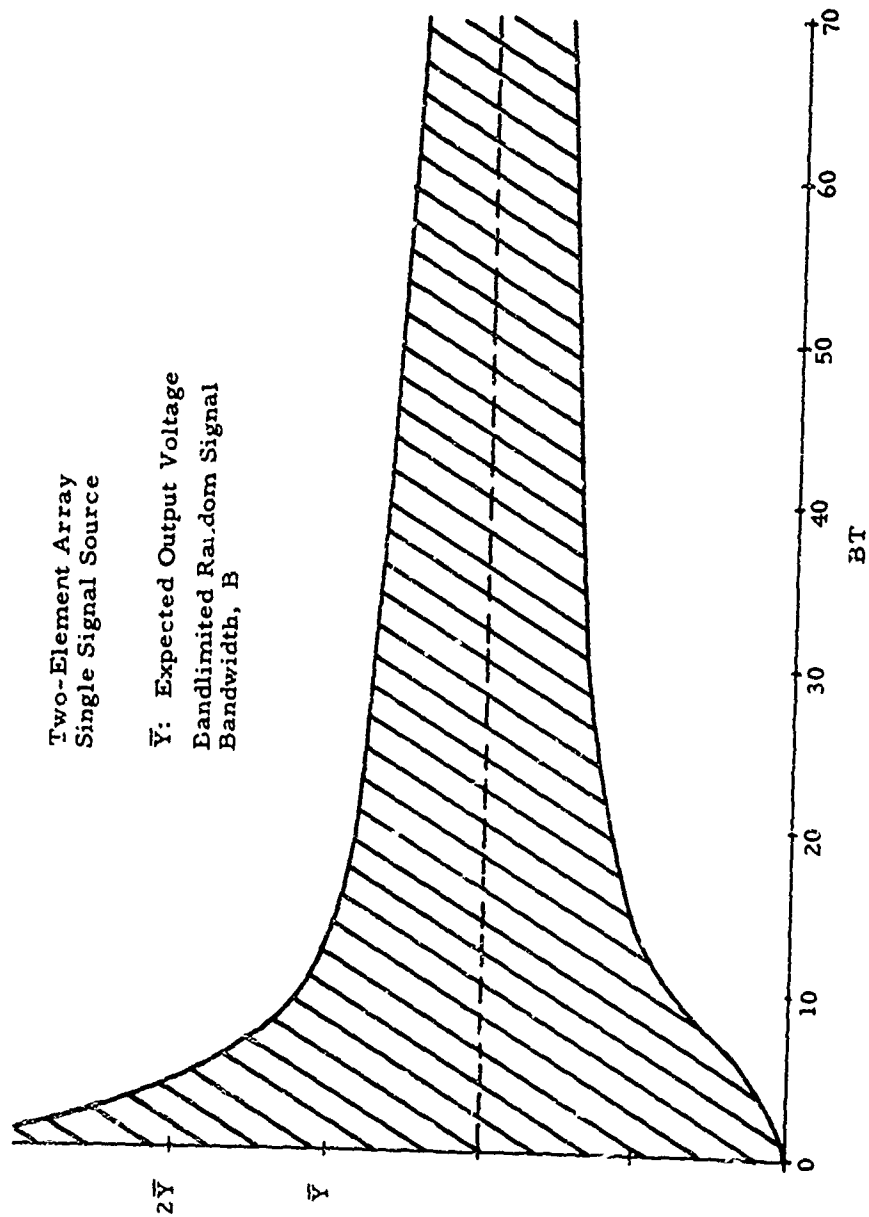
FIG. 5.—Ninety-five per cent confidence interval as a function of integration time.

this double frequency component of the product voltage, the output
voltage would fluctuate with only the low frequency component of vari-
ance. Regardless of this circuitry, however, the integration time that
will be necessary to reduce the fluctuation will be very long compared
to $1/f_0$, and the double frequency component of variance can generally
be completely neglected.

Incidentally, this statistical description of the fluctuation of the
output voltage of a two-element correlation array is the same as the
statistical description of the signal power output of a linear additive
array. The only difference would occur in the magnitude of the expected
value, $\overline{M_Y(T)}$. Knowing the variance of the linear array power fluc-
tuation as a function of time, we can compare averaging time require-
ments of more complex correlation arrays with that of the familiar
additive array.

The autocorrelation coefficient of the product voltage. $Y(t)$, for
general correlation arrays will be calculated in the same manner as
for the two-element array. The expected value of the product voltage
can be determined from the characteristic function as in the preceding
section. The autocorrelation coefficient of the product voltage for a
source located on the main lobe axis can also be determined by the
characteristic function method. This will be modified by inserting the
expected value of the arbitrarily located source, and the resulting form
will be used to approximate the actual autocorrelation coefficient.

The expected value of product voltage is given in the following
table for a few values of the number of elements, n.

n	$Y(t)$
2	$R(\tau_{12})$
4	$R(\tau_{12})R(\tau_{34}) + R(\tau_{13})R(\tau_{24}) + R(\tau_{14})R(\tau_{23})$
6	$\displaystyle\sum_{m,n,o,p,q=2}^{6} R(\tau_{1m})R(\tau_{no})R(\tau_{pq})$, $m \neq n \neq o \neq p \neq q$

The characteristic function for the product voltage is

$$F(\xi_1, \xi_2, s) = E\left\{\exp\left[j\xi_1 V(t) + j\xi_2 V(t+s)\right]\right\}$$

$$= \exp\left[-\frac{R(0)}{2}(\xi_1^2 + \xi_2^2) - R(s)\,\xi_1\xi_2\right]$$

Then $\psi_n(s)$ is the coefficient of $(j\xi_1)^n/n! \cdot (j\xi_2)^n/n!$ in the power series expansion of $F(\xi_1, \xi_2, s)$.

n	$\psi_n(s)$
2	$\overline{Y(t)}^2\,[1 + 2\,\rho(s)^2]$
4	$\overline{Y(t)}^2\,[9 + 72\,\rho(s)^2 + 24\,\rho(s)^4]$
6	$\overline{Y(t)}^2\,[225 + 4050\,\rho(s)^2 + 5400\,\rho(s)^4 + 720\,\rho(s)^6]$

Calculation of the variance, for the usual rectangular band-limited voltage, gives

for two elements (and for signal power in linear array) $\sigma_Y^2 = \overline{Y(t)}^2\,\dfrac{1}{BT}$

for four elements $\sigma_Y^2 = \overline{Y(t)}^2\,\dfrac{4.68}{BT}$

for six elements $\sigma_Y^2 = \overline{Y(t)}^2\,\dfrac{15.52}{BT}$

in all cases, $BT \gg 1$

This last table shows the general relationship between variance and number of elements in a correlation array. For any established level of fluctuation in the output voltage, a six-element correlation array will require an integration time 15.5 times as long as is required for a linear array (or a two-element correlation array). In the previous section the directivity pattern of a correlation array was found to be similar to that of a much longer linear array. Thus the correlation process in antennas is seen to be essentially one of trading space for time.

V. RESOLUTION CHARACTERISTICS OF CORRELATION ARRAYS

In describing the capability of a correlation array to resolve two similar signal sources separated by an angular displacement θ, the directivity patterns developed in Section III cannot be applied directly. Since the correlation arrays depend on multiplicative operations, the presence of two sources causes cross-product terms to appear. These terms then necessitate additional calculations to determine resolution capability. These calculations will be examined for correlation arrays in the presence of single frequency sources and randomly varying sources. The correlation arrays will be found to have resolution capabilities equivalent to those of longer linear arrays, if the sources to be resolved are independent; however, for coherent sources resolution may be possible only for certain specific separations of the sources and not as a general rule.

First, let us examine a four-element correlation array as developed in Section III. Assume there are two monochromatic sources, source A ($V_A = A \cos \omega_A t$) and source B ($V_B = B \cos \omega_B t$), separated by an angular displacement θ°. The general expression for the voltage response of the array as a function of position of the sources is quite lengthy, and is developed in Appendix I. It is found to depend on the coherence of the sources. The two cases that occur can be summarized:

1. For two coherent sources:

$$V_{out} \propto A^4(\cos X + \cos 3X + \cos 5X) + B^4(\cos Y + \cos 3Y + \cos 5Y)$$

$$+ A^3B \, f[\cos(nX-mY)] + A^2B^2 \, f[\cos(nX-mY)] + AB^3 f[\cos(nX-mY)]$$

2. For two sources of slightly different frequencies:

$$V_{out} \propto A^4(\cos X + \cos 3X + \cos 5X) + B^4(\cos Y + \cos 3Y + \cos 5Y)$$

$$+ A^3B \, f[\cos(nX-mY) \cdot \cos \delta t] + AB^3 f[\cos(nX-mY) \cdot \cos \delta t]$$

$$+ A^2B^2 \, f[\cos(nX-mY) \cdot \cos \delta t] + A^2B^2 \, f[\cos(nX-mY)]$$

where $X = (\omega D/c) \sin \phi_A$ and $Y = (\omega D/c) \sin \phi_B$ are the phase delay
factors for the two sources with element spacing D, and f[cos(nX-mY)]
represents a number of cosine terms containing these phase delay
factors, while f[cos (nX-mY)·cos δt] represents a number of these
cosine terms modified by the beat frequency component, δt, between
source A and source B.

In both of these equations, the first two terms correspond to
the voltage response pattern of a linear antenna array in the presence
of two sources. The remaining terms arise from the nonlinearity of
the correlation array. In the case of two sources with identical fre-
quencies, the cross-product terms are constant with time and averaging
the output voltage will not alter the result. When the sources have
different frequencies, however, time averaging can be employed to
reduce the heat frequency (cos δt) part of the cross-product terms;
but even in this case the resulting voltage response pattern will not be
precisely equivalent to that of the linear array, since some of the cross-
product terms will remain constant with time and will contribute to the
dc component of the output voltage.

The voltage response pattern of this four-element correlation
array as the array is rotated past a pair of coherent signal sources
separated by an angular displacement of 20° is shown in Fig. 6. In
this case the cross-products result in an apparent indication of a single
source located midway between the actual pair of sources. This four-
element correlation array will resolve sources with an angular separation
of 14° and with an angular separation of 28°, but a pair of sources whose
angular separation is in the range 0°-13° or in the range 15°-26° will
not be resolved by the array.

It should be noted that if the correlator contains only one stage
of multiplication, then the cross-product terms occurring from two
sources of different frequencies appear only as beat frequency components.
These terms describe a low frequency ac signal appearing with the
desired dc measurement voltage. In this case the cross-product terms
can be minimized by time averaging the output voltage. However, if
more than one stage of multiplication occurs between the antenna element
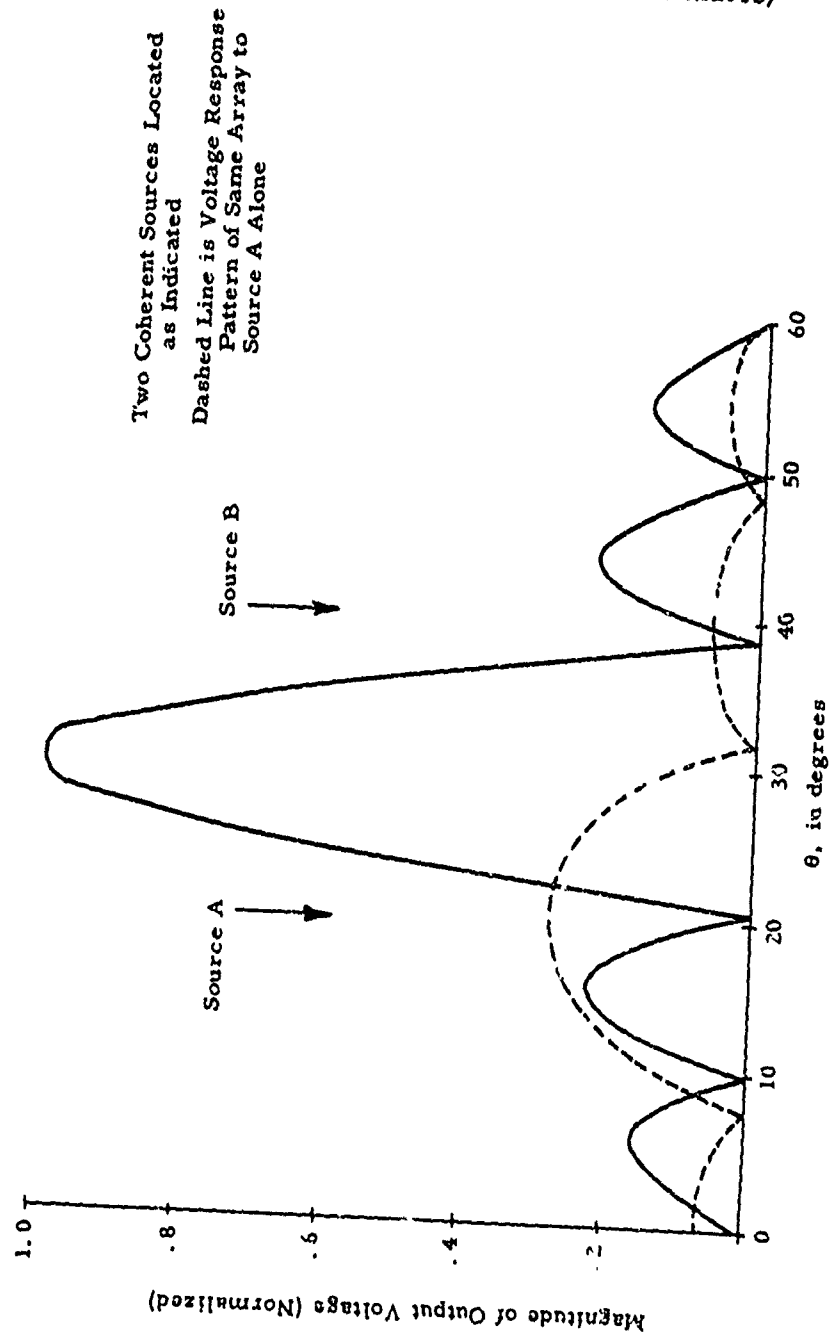
FIG. 6.—Voltage response pattern, four-element correlation array.

and the output of the array (as is the case in the example of the four-element correlation array), the cross-product terms will occur as low frequency ac terms and also as dc terms. In this case the equivalence between the correlation arrays and the linear arrays is not immediately apparent but must be determined by calculation of resolution characteristics for the particular type of signal encountered.

Assuming a basic spacing $D = \lambda/2$, the four-element correlation array of length 4D will resolve two sources of slightly different frequencies at approximately $19.5°$. This is equivalent to the resolution of a uniform linear array with an aperture of 9D, about twice as long.

A slightly more complex example of a correlation array is one discussed by Drane:[8]
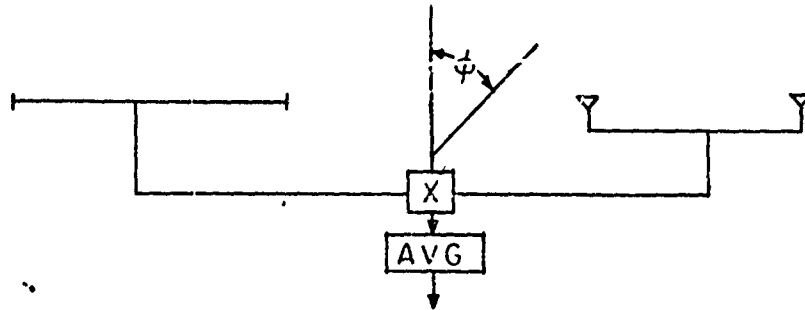
The element on the left represents a uniform linear array with aperture 2a, while that on the right is a simple interferometer with aperture 2b.

The directivity pattern of the uniform linear array is proportional to (sin X)/X, where $X = \omega(a/c) \sin \phi$. The directivity pattern of the interferometer is proportional to cos Y, where $Y = \omega(b/c) \sin \phi$. If a = b and there is no spacing between the right end of the linear array and the left element of the interferometer, the directivity pattern of the correlation array is

$$V_{out} \propto \frac{\sin 4X}{4X}$$

which is the same as the directivity pattern of a uniform linear array
of twice the length of this correlation array.

However, if two coherent sources are present the output becomes

$$V_{out} \propto \frac{\sin 4X_A}{4X_A} + \frac{\sin 4X_B}{4X_B} + \cos(X_A + X_B)\left[\frac{\sin X_A \cos X_B}{4X_A} + \frac{\cos X_A \sin X_B}{4X_B}\right]$$

The third term again occurs because of the nonlinearity of the correlator.
Calculation of the resolution capability of this array shows that it is
equivalent to a uniform linear array 1-1/2 times as long as the cor-
relation array.

These have been only two examples of the effect of cross-product
terms in a correlation array in the presence of monochromatic sources.
In each case that might be considered, the effect of these terms would
vary, depending on the types of signal voltages emitted by the sources
and by the number of successive multiplicative processes in the cor-
relator.

When two statistically similar sources are inducing randomly
varying voltages on the antenna elements, the resolution capabilities
of the correlation array must be expressed in terms of integration
times and confidence intervals. Consider again the four-element cor-
relation array and two independent sources (source A and source B).
Then the expected value of the output voltage can be calculated either
directly or by the characteristic function method and can be expressed
in terms of the correlation coefficients of the individual voltages. The
voltage induced on the i-th element is

$$V_i(t) = V_A(t + \tau_i) + V_B(T + \tau_i)$$

and

$$\overline{Y(t)} = \overline{V_1(t)V_2(t)V_3(t)V_4(t)} =$$
$$\{ [R_A(\tau_{12})R_A(\tau_{34}) + R_A(\tau_{13})R_A(\tau_{24}) + R_A(\tau_{14})R_A(\tau_{23})]$$
$$+ [R_B(\tau_{12})R_B(\tau_{34}) + R_B(\tau_{13})R_B(\tau_{24}) + R_B(\tau_{14})R_B(\tau_{23})]$$
$$+ [R_A(\tau_{12})R_B(\tau_{34}) + R_A(\tau_{13})R_B(\tau_{24}) + R_A(\tau_{14})R_B(\tau_{23})$$
$$+ R_A(\tau_{34})R_B(\tau_{12}) + R_A(\tau_{24})R_A(\tau_{13}) + R_A(\tau_{23})R_B(\tau_{14}) \}$$

The first and second lines in this expression give the expected voltages due to each source individually. The third and fourth lines contain the cross-product components which occur as a result of the two stages of multiplication of the element voltages.

The variance of this output voltage could be calculated directly; however, the large number of terms in the final expression would make this quite laborious. Therefore, this quantity will be approximated as previously by considering the variance of the voltage produced by a single source on the main lobe axis and then introducing the new value of $\overline{Y(t)}$ given by the equation above.

With an arbitrary angular displacement between the two sources, and a given level of confidence desired in the result, it is possible to calculate the maximum variance which will permit a significant difference in the amplitude of the output voltage when the array is directed at a source and when it is directed between the sources. This value of variance can then be used to determine the minimum permissible integration time. So the array resolution is a function of two quantities: level of confidence and integration time.

The resolution capabilities of the simple four-element correlation array are listed in the following table. They are compared with those of a linear array.

<div align="center">

Random Signals

Resolution at the 95 Per Cent Confidence Level

</div>

Four-Element Correlation Array Length 4D		Aperture of Equivalent Uniform Linear Array
Resolution	Integration Time	
19°	BT = 18	9.6 D
18°	BT = 97	9.4 D
17.5°	BT = 890	9.2 D
17°	BT = ∞	

VI. SUMMARY

Consideration of antenna arrays designed to receive randomly varying signals buried in a noisy medium is facilitated by the use of statistical correlation techniques. With the familiar linear additive array, the correlation coefficients of signal and of spatially distributed noise can be used to obtain the element spacing for optimum signal reception. With nonlinear arrays the output voltage again is expressed in terms of the correlation coefficients of the signal and noise voltages induced on the elements.

The directivity patterns obtained for the nonlinear arrays were shown to be equivalent to those of linear arrays of greater length. These patterns, and an analysis of the averaging time required to perform the correlation process, show that the correlation technique in antennas is an exchange of length for time.

When more than one signal source is present, the correlation array encounters difficulties from the cross-product terms occurring in the multiplication processes. With two coherent sources, resolution may only be possible at specific separations, rather than over a general range of separations. In this case, each array must be examined under the actual signal source conditions to determine exact behavior. With independent sources, the process of correlation reduces the cross-product effect, and the principle of length-time exchange is again valid.

This is the second report in this general study of correlation techniques in antenna systems. Future work will be concerned with a study of correlation arrays in the presence of distributed noise, and with a determination of conditions of signals under which correlation arrays are superior to linear arrays.

APPENDIX I: CALCULATION OF VOLTAGE
RESPONSE PATTERNS WITH TWO SIGNAL SOURCES

The antenna will be a four-element correlation array with relative spacings $d_{12} = D$, $d_{23} = D$, $d_{34} = 2D$, and $D = \lambda/2$. The voltage
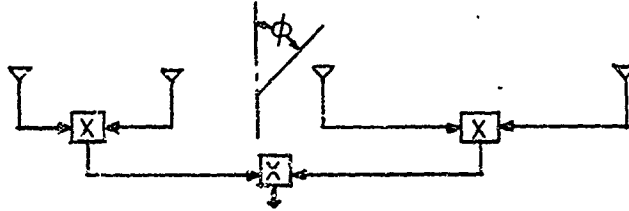
induced on the i-th element by signal source A is $V_A = A \cos \omega_A(t + \tau_{Ai})$ and that by signal source B is $V_B = B \cos \omega_B(t + \tau_{Bi})$, where $\tau_{Ai} = (di/c) \sin \phi_B$. The signals which are interesting in this study of correlation processes are grouped into two cases: (1) coherent signals ($\omega_A = \omega_B$) and (2) signals with slightly different frequencies ($\omega_B - \omega_A = \delta\omega$). Therefore, $(\omega_A d_i/c) \approx (\omega_B d_i/c)$ and the following notation can be used:

$$X = \frac{\omega D}{c} \sin \phi_A \qquad \text{and} \qquad Y = \frac{\omega D}{c} \sin \phi_B$$

Then the product of the four-element voltages can be written:

$$V = \{[A \cos \omega_A t + B \cos \omega_B t][A \cos(\omega_A t + X) + B \cos(\omega_B t + Y)] \cdot$$

$$\cdot [A \cos(\omega_A t + 2X) + B \cos(\omega_B t + 2Y)][A \cos(\omega_A t + 4X) + B \cos(\omega_B t + 4Y)]\}$$

Carrying out the indicated multiplications, and rejecting the multiple frequency components (2ω and 4ω), gives

$$V = \frac{A^4}{8}[\cos X + \cos 3X + \cos 5X] + \frac{B^4}{8}[\cos Y + \cos 3Y + \cos 5Y]$$

$$+ \frac{A^3B}{8}[\cos(X+\delta\omega t)+\cos(3X-\delta\omega t)+\cos(5X+\delta\omega t)+\cos(X+4Y+\delta\omega t)$$
$$+ \cos(X-4Y-\delta\omega t)+\cos(2X+Y+\delta\omega t)+\cos(2X-Y-\delta\omega t)+\cos(3X+2Y+\delta\omega t)$$
$$+ \cos(3X-2Y-\delta\omega t)+\cos(3X-4Y-\delta\omega t)+\cos(5X-2Y-\delta\omega t)+\cos(6X-Y-\delta\omega t)]$$

$$+ \frac{AB^3}{8}[\cos(Y-\delta\omega t)+\cos(3Y+\delta\omega t)+\cos(5Y+\delta\omega t)+\cos(Y+4X-\delta\omega t)$$
$$+ \cos(Y-4X+\delta\omega t)+\cos(2Y+X-\delta\omega t)+\cos(2Y-X+\delta\omega t)+\cos(3Y+2X-\delta\omega t)$$
$$+ \cos(3Y-2X+\delta\omega t)-\cos(3Y-4X+\delta\omega t)+\cos(5Y-2X+\delta\omega t)+\cos(6Y-X+\delta\omega t)]$$

$$+ \frac{A^2B^2}{8}[\cos(X+2Y)+\cos(X-2Y)+\cos(X+4Y)+\cos(X-4Y)+\cos(2X+Y)+\cos(2X-Y)$$
$$+ \cos(2X+3Y)+\cos(2X-3Y)+\cos(3X+2Y)+\cos(X-2Y)+\cos(4X+Y)$$
$$+ \cos(4X-Y)+\cos(X-6Y-2\delta\omega t)+\cos(2X-5Y-2\delta\omega t)+\cos(3X-4Y-2\delta\omega t)$$
$$+ \cos(4X-3Y-2\delta\omega t)+\cos(5X-2Y-2\delta\omega t)+\cos(6X-Y-2\delta\omega t)]$$

Although this is a formidable expression, the capability of the array to resolve the two sources can be estimated by examining two simplified equations:

Case 1... when source A is located on the principal lobe of the array (X = 0),

Case 2... when the principal lobe of the array is directed between the two sources (X = Y).

If it is assumed that A = B and that $\delta\omega = 0$, then the equations for these two cases are:

Case 1
$$V = \frac{A^4}{8}[6 + 12\cos Z' + 10\cos 2Z' + 8\cos 3Z' + 6\cos 4Z'$$
$$+ 4\cos 5Z' + 2\cos 6Z']$$

Case 2
$$V = \frac{A^4}{8}[12\cos Z'' + 12\cos 3Z'' + 12\cos 5Z'' + 12\cos 7Z'']$$

where, with the angular displacement of the sources denoted by θ,

$$Z' = \pi \sin \theta, \quad \text{and} \quad Z'' = \pi \sin \frac{\theta}{2}$$

Resolution of the two sources is certainly possible when the magnitude of the voltage response with the main lobe directed between the sources (case 2) is less than the magnitude of the voltage response with the main lobe directed at one of the sources (case 1). Although this is not the only condition under which resolution will occur, a comparison of these two voltage responses as functions of the separation of the sources should indicate the general action of the array in the presence of two sources. Resolution is certain for separations of approximately
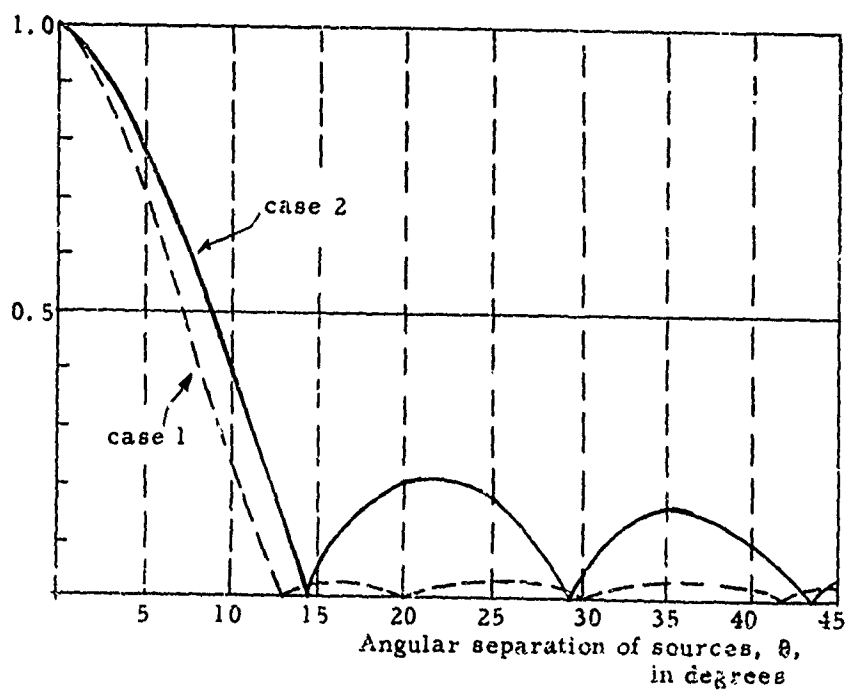
VOLTAGE RESPONSE OF ARRAY AS A FUNCTION OF SEPARATION OF SOURCES.
Dashed line: main lobe directed at one source.
Solid line: main lobe directed between sources.

14^0, 29^0, and 43^0 where the voltage response for the main lobe directed between the sources drops to zero. It is interesting to note that the voltage response when the main lobe is directed at one of the sources is uniformly low, and the array will not indicate a maximum voltage response for this orientation. The general voltage response patterns of this array for several specific source separations are shown in the accompanying diagrams, (Figs. 7, 8 and 9).
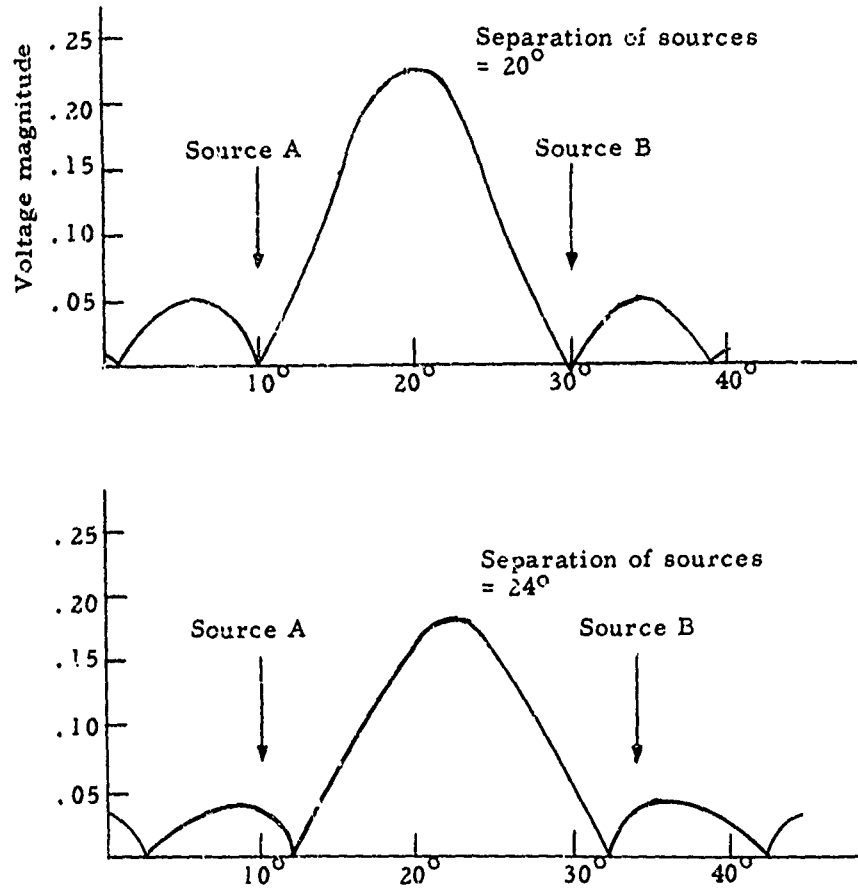
FIG. 7.—Voltage response pattern, four-element correlation array and two coherent signal sources. Voltage normalized to unity for maximum response of two coincident signal sources.
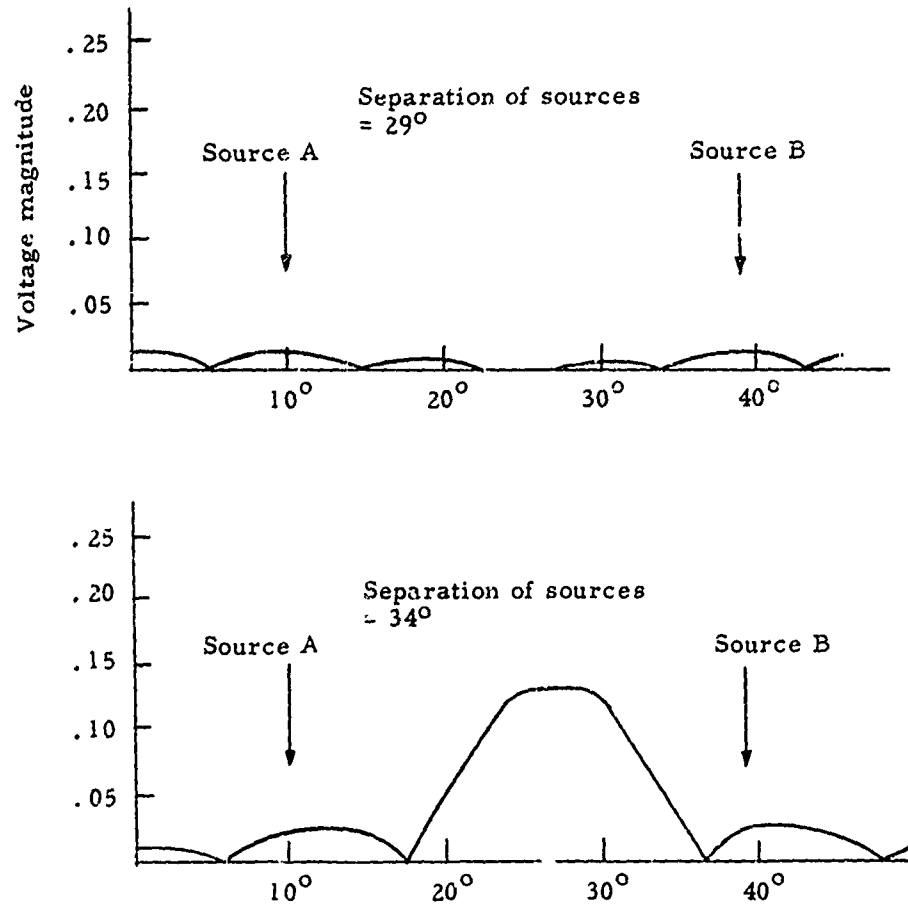
FIG. 8.—Voltage response pattern, four-element correlation array and two coherent signal sources. Voltage normalized to unity for maximum response of two coincident signal sources.
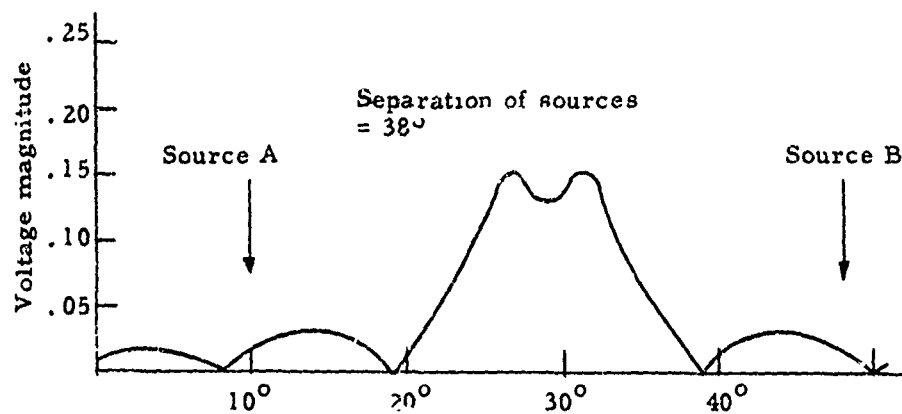
FIG. 9.—Voltage response pattern, four-element correlation array and two coherent signal sources. Voltage normalized to unity for maximum response of two coincident signal sources.

A similar response pattern, resolving the two sources, can be obtained with a linear antenna with aperture 3λ (6D). This is approximately the same length as the correlation array whose patterns are plotted here.

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